Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Control of Distributed Structures with Small Nonproportional Damping

L. Meirovitch* and M. A. Norris†
Virginia Polytechnic Institute and State University, Blacksburg, Virginia

Introduction

STRUCTURES are distributed-parameter systems characterized by self-adjoint stiffness operators.[1] In the absence of damping, the structures themselves are self-adjoint, so that the eigenfunctions are orthogonal. In general, damping tends to destroy the self-adjointness property.

The self-adjointness property is very important in control design. Indeed, for undamped structures it is possible to describe the system by a set of independent *modal equations.* Then, controls can be designed for each modal equation independently.[2]

This Note examines the circumstances under which controls based on a self-adjoint structure yield satisfactory performance when applied to a closely related nonself-adjoint structure. To this end, a perturbation technique is used to compute the sensitivity of the closed-loop poles to small viscous damping. A numerical example is presented in which the control system performance is examined for varying amounts of nonproportional damping. The example indicates how large the nonproportional damping can be without degrading control system performance.

Equations of Motion

The motion of a distributed structure can be described by the partial differential equation[1]

$$m(P)\ddot{u}(P,t) + \mathcal{C}\dot{u}(P,t) + \mathcal{L}u(P,t) = f(P,t), \quad P\epsilon D \tag{1}$$

where $u(P,t)$ is the displacement at point P in the domain D of the structrure, $m(P)$ the mass distribution, $\mathcal{C}$ a viscous damping operator, $\mathcal{L}$ a self-adjoint differential operator of order $2p$ expressing the system striffness, where p is an integer, and $f(P,t)$ the external force density. The displacement $u(P,t)$ is subject to the boundary conditions $B_i u(P,t) = 0$, $P\epsilon S(i=1,2,\ldots,p)$, where B_i are differential operators of maximum order $2p-1$, and S defines the boundary of D.

We examine the case in which the term $\mathcal{G}\dot{u}(P,t)$ is of one order of magnitude smaller than the remaining terms in Eq. (1), and consider the eigenvalue problem associated with the undamped self-adjoint structure, or $\mathcal{L}\phi_r(P) = \lambda_r m(P)\phi_r(P)$ $(r=1,2,\ldots)$. The solution of the eigenvalue problem consists of a denumerably infinite set of real nonnegative eigenvalues λ_r and associated real eigenfunctions $\phi_r(P)$, where $\phi_r(P)$ are subject to suitable boundary conditions. The eigenvalues are related to the natural frequencies of undamped vibration by $\lambda_r = \omega_r^2$ $(r=1,2,\ldots)$. The eigenfunctions can be normalized so as to satisfy

$$\int_D m\phi_r(P)\phi_s(P)\,\mathrm{d}D = \delta_{rs}$$

$$\int_D \phi_r(P)\mathcal{L}\phi_s(P)\,\mathrm{d}D = \lambda_r\delta_{rs}, \qquad r,s = 1,2,\ldots$$

where δ_{rs} is the Kronecker delta. We propose to represent the solution of Eq. (1) by the infinite series

$$u(P,t) = \sum_{r=1}^{\infty} \phi_r(P)u_r(t) \tag{2}$$

where $u_r(t)$ are time-dependent generalized coordinates. Introducing Eq. (2) into Eq. (1), multiplying the result by ϕ_s and integrating over the domain D, we obtain

$$\ddot{u}_r(t) + \sum_{s=1}^{\infty} c_{rs}\dot{u}_s(t) + \omega_r^2 u_r(t) = f_r(t), \qquad r=1,2,\ldots \tag{3}$$

where

$$c_{rs} = \int_D \phi_r \mathcal{C}\phi_s\,\mathrm{d}D, \qquad r,s=1,2,\ldots \tag{4}$$

are viscous damping coefficients, and

$$f_r(t) = \int_D \phi_r(P)f(P,t)\,\mathrm{d}D, \qquad r=1,2,\ldots$$

are generalized control forces.

In general, the matrix $C=[c_{rs}]$ is fully populated. In this case, to obtain a solution, it is necessary to cast Eqs. (3) in the state space, which can be written in the vector form

$$\dot{x} = A^*x + B^*f \tag{5}$$

where $x(t) = [u^T(t) \,|\, \dot{u}^T(t)]^T$ is the state vector, and

$$A^* \left[\begin{array}{c|c} 0 & I \\ \hline -\Lambda & -C \end{array}\right], \qquad B^* = \left[\begin{array}{c} 0 \\ \hline I \end{array}\right] \tag{6}$$

are coefficient matrices in which I is the identity matrix and $\Lambda = \mathrm{diag}(\omega_r^2)$ $(r=1,2,\ldots)$. Of course, $u(t)$ and $f(t)$ are the vectors of generalized coordinates and generalized forces, respectively.

Sensitivity Analysis

We consider the case in which the control law is given by

$$f(t) = -Gu(t) - H\dot{u}(t) \tag{7}$$

where G, H are control gain matrices. Substituting Eq. (7) into Eq. (5), we obtain the closed-loop state equations of motion in the vector form

$$\dot{x} = Ax \tag{8}$$

where A is the closed-loop coefficient matrix.

Received Jan. 12, 1988; revision received Sept. 5, 1988.

*University Distinguished Professor, Department of Engineering Science and Mechanics. Fellow AIAA.

†Assistant Professor, Department of Engineering Science and Mechanics. Member AIAA.

When damping in the actual distributed structure is small, i.e., when $c_{rs} \to 0$ $(r,s=1,2,\ldots)$, the matrix A can be expressed in the perturbed form

$$A = A_0 + A_1 \tag{9}$$

where

$$A_0 = \left[\begin{array}{c|c} 0 & I \\ \hline -\Lambda - G & -H \end{array}\right] \tag{10a}$$

$$A_1 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & -C \end{array}\right] \tag{10b}$$

in which A_1 is "small" relative to A_0. Because A is obtained from A_0 by a small perturbation, we assume that the eigensolutions can be written in the form

$$\lambda_r = \lambda_{0r} + \lambda_{1r} \tag{11a}$$

$$u_r = u_{0r} + u_{1r} \tag{11b}$$

$$v_r = v_{0r} + v_{1r} \tag{11c}$$

where the subscripts 0 and 1 denote unperturbed eigensolutions and first-order perturbations, respectively. It can be shown that the eigenvalue perturbations have the expressions[1]

$$\lambda_{1r} = v_{0r}^T A_1 u_{0r}, \qquad r=1,2,\ldots \tag{12}$$

Equations (12) can be regarded as providing the sensitivity of the closed-loop poles to small viscous damping in the distributed structure. In general, the closed-loop right and left eigenvectors u_{0r} and v_{0r} $(r=1,2,\ldots)$ are fully populated, so that the perturbations λ_{1r} $(r=1,2,\ldots)$ are sensitive to all the damping coefficients c_{rs} $(r,s=1,2,\ldots)$.

Sensitivity Using Natural Control

Control of self-adjoint structures can be carried out by natural control, which is characterized by the preservation of the natural coordinates in the closed-loop system, owing to the fact that the open-loop eigenfunctions are identical to the closed-loop eigenfunctions. Natural controls can be obtained by the independent modal space control (IMSC) method.[2] In the IMSC method, the control gain matrices G and H are diagonal, with the diagonal entries equal to g_r and h_r $(r=1,2,\ldots)$, respectively.

It can be shown that in IMSC the right and left eigenvectors of the zero-order problem have the expression

$$u_{0r} = \left[\begin{array}{c} e_r \\ \hline \lambda_{0r} e_r \end{array}\right], \qquad v_{0r} = \frac{1}{\alpha_r}\left[\begin{array}{c} (\omega_r^2 + g_r) e_r \\ \hline -\lambda_{0r} e_r \end{array}\right], \qquad r=1,2,\ldots \tag{13}$$

where e_r is a standard unit vector (with all its components equal to zero, with the exception of the rth component, which is equal to 1), and $\alpha_r = \omega_r^2 + g_r - \lambda_{0r}^2$ are normalization constants. Hence, in the case of IMSC, every one of the closed-loop eigenvectors u_{0r}, v_{0r} $(r=1,2,\ldots)$ associated with the undamped structure contain only two nonzero terms. The closed-loop eigenvalues λ_{0r} have the expressions

$$\lambda_{0r} = -\tfrac{1}{2}\{h_r + i[4(\omega_r^2 + g_r) - h_r^2]^{1/2}\}, \quad r=1,2,\ldots \tag{14}$$

Equations (13) and (14) specify only one-half of the eigenvalues and left and right eigenvectors. The complex conjugates comprise the other half.

Inserting Eqs. (10b), and (13) into Eq. (12), we can write

$$\lambda_{1r} = (\lambda_0^2/\alpha_r)c_{rr}, \qquad r=1,2,\ldots \tag{15}$$

where the real part of λ_{0r}^2/α_r is always negative. Examining Eqs. (15), it is evident that the perturbations in the closed-loop eigenvalues are functions only of the diagonal terms in the matrix C. Hence, damping perturbations in which $c_{rr} \geq 0$ $(r=1,2,\ldots)$ can only shift the actual structure closed-loop poles to the left in the complex plane relative to the corresponding closed-loop poles of the undamped structure.

Numerical Example

As an illustration, we consider the control of the bending vibration of a simply supported beam. For uniform mass and stiffness distributions, Eq. (1) becomes[3]

$$M\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[cI(x)\frac{\partial^3 u}{\partial x^2 \partial t^2}\right] + EI\frac{\partial^4 u}{\partial x^4} = f, \qquad 0<x<L \tag{16}$$

where we note that the damping operator

$$\mathcal{C} = (\partial^2/\partial x^2)\,[cI(x)(\partial^2/\partial x^2)]$$

provides strain-rate damping, which is a form of viscous damping. The displacement u is subject to the boundary conditions $u(0,t) = \partial^2 u(0,t)/\partial x^2 = u(L,t) = \partial^2 u(L,t)/\partial x^2 = 0$. For simplicity, we choose $M=1$, $EI=1$, and $L=5$.

It can be shown that in the undamped case, i.e., for $cI(x)=0$, the eigensolution consists of the eigenfunctions and natural frequencies[1]

$$\phi_r(x) = \left(\frac{2}{L}\right)^{1/2}\sin\left(\frac{r\pi x}{L}\right), \qquad \omega_r = \left(\frac{r\pi}{L}\right)^2, \qquad r=1,2,\ldots \tag{17}$$

respectively. For the damped case, we choose the strain-rate damping distribution $cI(x) = \alpha + \beta x(L-x)$ $(0<x<L)$, where α and β are constants. The damping coefficients in Eq. (4) can be computed in closed form.

We consider controlling a subset of the modes using IMSC. More specifically, the first three modes are to be controlled using three discrete actuators at $x_i = iL/4$ $(i=1,2,3)$. To determine the control gain matrices G and H in Eq. (7), we minimize the performance index

$$J = \sum_{r=1}^{3}\int_0^\infty \{\dot{u}_r^2(t) + \lambda_r u_r^2(t) + R_r f_r^2(t)\}\,dt \tag{18}$$

The minimization provides diagonal control gain matrices.[2]

To show the effects of damping, we investigated three cases, 1, 2, and 3, corresponding to $\alpha=\beta=0$, $\alpha=\beta=0.0001$, and $\alpha=\beta=0.001$, respectively. Table 1 displays the closed-loop poles for the three cases with $R_r=1.0$ $(r=1,2,3)$. Moreover, the shift in the closed-loop poles for the controlled modes, i.e., the lowest three modes, can be computed using Eqs. (14) and (15).

To examine the control system performance, we considered cases 1, 2, and 3, except that we increased the control effort by decreasing R_r in Eq. (18). For $R_r=1.0$ $(r=1,2,3)$, damping enhances the control system performance, as the performance index $J(t)$ at $t=5$ s has the values 893.1, 883.7, and 813.1 corresponding to cases 1, 2, and 3, respectively. For $R_r=0.01$ $(r=1,2,3)$, we obtained the opposite effect, however, with the performance index $J(t)$ at $t=5$ s having the values 250.3,

Table 1 Closed-loop poles for $R_r = 1.0$ ($r = 1,2,3$)

Case 1 $\alpha=\beta=0$	Case 2 $\alpha=\beta=0.0001$	Case 3 $\alpha=\beta=0.00$
$-0.6199 \pm i1.0057$	$-0.6200 \pm i1.0056$	$-0.6211 \pm i1.0050$
$-0.6921 \pm i1.8436$	$-0.6936 \pm i1.8430$	$-0.7076 \pm i1.8382$
$-0.7037 \pm i3.6792$	$-0.7112 \pm i3.6778$	$-0.7815 \pm i3.6676$
$\pm i6.3165$	$-0.0231 \pm i6.3168$	$-0.2248 \pm i6.3379$
$\pm i9.8696$	$-0.0562 \pm i9.8704$	$-0.5643 \pm i9.9485$
$\pm i14.2122$	$-0.1162 \pm i14.2146$	$-1.1630 \pm i14.4489$
$\pm i19.3444$	$-0.2148 \pm i19.3504$	$-1.8443 \pm i20.4288$
$\pm i25.2662$	$-0.3360 \pm i25.2453$	$-3.9635 \pm i22.6735$

250.4, and 252.3 corresponding to cases 1, 2, and 3, respectively. Hence, damping may enhance or degrade control system performance depending on the value of the gains. As the gains increase, larger amounts of energy are transferred back and forth between the controlled and uncontrolled modes, which are coupled due to the presence of damping. This effect is known as control spillover. However, we also conclude that the damping does not alter control system performance significantly.

Conclusions

Quite often, control system design is based on the undamped self-adjoint distributed structure. Inherent structural damping tends to destroy the self-adjointness of the structure and can degrade control system performance. However, depending on the control system design, the effect of small damping can be negligible. Indeed, when control is designed by the independent modal space control method (IMSC), the perturbation in the closed-loop poles due to small damping does not cause a dramatic shift in the closed-loop poles nor does it change the control system performance very much. The sensitivity study presented here for IMSC indicates that damping shifts the actual closed-loop poles to the left of the modeled closed-loop poles in the complex plane. Moreover, in the case of IMSC, the presence of small damping in the actual distributed structure does not affect the control system performance significantly.

Acknowledgments

Supported in part by the USAF/ASD and the Air Force Office of Scientific Research Grant F33615-86-C-3233, monitored by Drs. A. K. Amos and V. B. Venkayya, whose support is greatly appreciated.

References

[1]Meirovitch, L., *Computational Methods in Structural Dynamics*, Sijthoff & Noordhoff, the Netherlands, 1980.

[2]Meirovitch, L. and Baruh, H., "Control of Self-Adjoint Distributed-Parameter Systems," *Journal of Guidance and Control*, Vol. 5, Jan.-Feb. 1982, pp. 60-66.

[3]Clough, R. W. and Penzien, J., *Dynamics of Structures*, McGraw-Hill, New York, 1975.

Application of Output Feedback to Variable Structure Systems

Bonnie S. Heck* and Aldo A. Ferri†
Georgia Institute of Technology, Atlanta, Georgia

I. Introduction

VARIABLE structure systems (VSS), especially those with a sliding mode, have generated considerable interest recently in control-theory applications due to the benefits achieved: good robustness, disturbance rejection, model reduction, and possible linearization through the control. This control scheme has been applied in chemical and industrial processes, flight control, spacecraft attitude control, robotics, and motor control.[1-4] More information and a literature survey may be found in the papers by Utkin.[1,5]

The use of VSS has been previously limited to systems with full-state feedback. In practice, however, full measurement of the state vector is either not possible or not feasible. Instead, only certain outputs are measured. Recently, asymptotic observers have been used in VSS to reconstruct the state vector for estimated state feedback.[6,7] This adds dynamics to the compensator, which increases the complexity in the implementation. In fact, it may be physically unrealizable for a large-order system. Also, convergence to the sliding manifold is asymptotic as $t \to +\infty$, and so the invariance properties found with state feedback do not strictly hold. Even if the states are all measurable, implementation with state feedback is necessarily complex, requiring many feedback loops. The purpose of this Note is to introduce output feedback as an alternative to state feedback and to estimated state feedback for simplification of control-system implementation.

Output feedback used in linear systems has generated much interest in the last 20 years (see, for example, Refs. 8-10). The main benefits are that there are no dynamics in the compensator and there are fewer feedback loops. There are, however, inherent problems with output feedback in linear systems that are also present when applied to VSS. For example, it may not be possible to stabilize a system with a given set of outputs using static output feedback. Also, output feedback is generally difficult to design. However, in many cases, the simplification in implementation far outweighs the difficulties in design.

The outline of this Note is as follows. Section II contains the basic considerations of output feedback in VSS. Section III discusses two design procedures. An example illustrating the technique is given in Sec. IV, and the conclusions are given in Sec. V.

II. Output Feedback in VSS

The system under consideration is time-invariant and can be represented in the following form

$$\dot{x} = f(x) + B(x)u \tag{1}$$

$$y = Cx \tag{2}$$

where $x \epsilon R^n$, $u \epsilon R^m$, $y \epsilon R^p$, f and B satisfy the Lipschitz conditions, and C is a matrix. [For simplicity of notation, $f(x)$ will be denoted as f, and $B(x)$ will be denoted as B.] The components of the control vector u are given as a function of the output y, i.e.,

$$u_i = \begin{cases} u_i^+(y) & \text{if } s_i(y)>0 \\ u_i^-(y) & \text{if } s_i(y)<0 \end{cases}, \qquad i = 1,\dots,m \tag{3}$$

where each s_i is a linear functional, and each u_i^+ and u_i^- satisfies a Lipschitz condition. A sliding mode may occur on any of the surfaces defined by $s_i(y) = 0$ or along any intersection of these surfaces. This Note will address the case of sliding along the surface $s(y) = [s_1,\dots,s_m]^T(y) = 0$, although the other cases are easily extracted from this analysis.

Using the equivalent control method,[1] the equation of motion for the system [Eqs. (1) and (2)] in the sliding mode is given by:

$$\dot{x} = f - B(GCB)^{-1}GCf \tag{4}$$

where $s = Gy$. Note that Eq. (4) is valid only when $s = 0$. The switching function $s = Gy$ is chosen to stabilize the sliding-mode equation under the constraint that GCB be invertible.

Received Jan. 8, 1988; revision received Aug. 1, 1988.

*Assistant Professor, School of Electrical Engineering.

†Assistant Professor, School of Mechanical Engineering. Member AIAA.